Ex 12B

1. If the number is even, we can represent it as $2n$, for a suitably chosen integer value n.

$$(2n)^2 = 4n^2$$

which is a multiple of 4, and hence is even.

If the number is odd, we can represent it as $2n+1$ for a suitably chosen integer value n.

$$\begin{aligned}(2n+1)^2 &= 4n^2 + 4n + 1\\ &= 4(n^2+n)+1\end{aligned}$$

which is one more than a multiple of 4, and hence is odd.

2. There are five possible cases. The number is

- a multiple of 5;
- one more than a multiple of 5;
- two more than a multiple of 5;
- three more than a multiple of 5; or
- four more than a multiple of 5.

Considering each of these exhaustively:

- If the number is a multiple of 5:

$$\begin{aligned}(5n)^2 &= 25n^2\\ &= 5(5n^2)\end{aligned}$$

The square is a multiple of 5.

- If the number is one more than a multiple of 5:

$$\begin{aligned}(5n+1)^2 &= 25n^2 + 10n + 1\\ &= 5(5n^2+2n)+1\end{aligned}$$

The square is 1 more than a multiple of 5.

- If the number is two more than a multiple of 5:

$$\begin{aligned}(5n+2)^2 &= 25n^2 + 20n + 4\\ &= 5(5n^2+4n)+4\end{aligned}$$

The square is 4 more than a multiple of 5.

- If the number is three more than a multiple of 5:

$$\begin{aligned}(5n+3)^2 &= 25n^2 + 30n + 9\\ &= 5(5n^2+4n+1)+4\end{aligned}$$

The square is 4 more than a multiple of 5.

- If the number is four more than a multiple of 5:

$$\begin{aligned}(5n+4)^2 &= 25n^2 + 80n + 16\\ &= 5(5n^2+4n+3)+1\end{aligned}$$

The square is 1 more than a multiple of 5.

3. The the number is

- a multiple of 3;
- one more than a multiple of 3; or
- two more than a multiple of 3.

Considering each of these exhaustively:

- If the number is a multiple of 3:

$$\begin{aligned}(3n)^3 &= 27n^3\\ &= 9(3n^3)\end{aligned}$$

The cube is a multiple of 9.

- If the number is one more than a multiple of 9:

$$\begin{aligned}(3n+1)^3 &= (9n^2+6n+1)(3n+1)\\ &= 27n^3 + 9n^2 + 18n^2 + 6n + 3n + 1\\ &= 27n^3 + 27n^2 + 9n + 1\\ &= 9(3n^3 + 3n^3 + n) + 1\end{aligned}$$

The cube is 1 more than a multiple of 9.

- If the number is two more than a multiple of 3:

$$\begin{aligned}(3n+2)^3 &= (9n^2+12n+4)(3n+2)\\ &= 27n^3 + 18n^2 + 36n^2 + 24n + 12n + 8\\ &= 27n^3 + 54n^2 + 36n + 9 - 1\\ &= 9(3n^3 + 6n^3 + 4n + 1) - 1\end{aligned}$$

The cube is 1 less than a multiple of 9.

4.
- Suppose T_n is even, i.e. $T_n = 2x$ for some integer x, then

$$\begin{aligned} T_{n+1} &= 3T_n + 2 \\ &= 3(2x) + 2 \\ &= 2(3x + 1) \end{aligned}$$

Hence T_{n+1} is also even.

- Suppose T_n is odd, i.e. $T_n = 2x+1$ for some integer x, then

$$\begin{aligned} T_{n+1} &= 3T_n + 2 \\ &= 3(2x + 1) + 2 \\ &= 6x + 5 \\ &= 2(3x + 2) + 1 \end{aligned}$$

Hence T_{n+1} is also odd.

$\therefore T_{n+1}$ has the same parity as T_n.

5. Consider $x^5 - x = x(x-1)(x+1)(x^2+1)$

- If $x = 5n$ then $x^5 - x$ has $x = 5n$ as a factor, so it is a multiple of 5.
- If $x = 5n+1$ then $x^5 - x$ has $(x-1) = (5n+1) - 1 = 5n$ as a factor, so it is a multiple of 5.
- If $x = 5n + 2$ then

$$\begin{aligned} x^2 + 1 &= (5n+2)^2 + 1 \\ &= 25n^2 + 20n + 5 \\ &= 5(5n^2 + 4n + 1) \end{aligned}$$

Hence as $x^5 - x$ has $(x^2+1) = 5(5n^2+4n+1)$ as a factor, it is a multiple of 5.

- If $x = 5n + 3$ then

$$\begin{aligned} x^2 + 1 &= (5n+3)^2 + 1 \\ &= 25n^2 + 30n + 10 \\ &= 5(5n^2 + 6n + 2) \end{aligned}$$

Hence as $x^5 - x$ has $(x^2+1) = 5(5n^2+6n+2)$ as a factor, it is a multiple of 5.

- If $x = 5n+4$ then $x^5 - x$ has $(x+1) = (5n+4)+1 = 5n+5 = 5(n+1)$ as a factor, so it is a multiple of 5.

Hence, $x^5 - x$ for $x > 1$ is always a multiple of 5.

As one or other of x and $x-1$ is even, $x^5 - x$ always has 2 as a factor. Since it has both 2 and 5 as factors, it is always a multiple of 10.

If x is odd, both $x-1$ and $x+1$ are even, so $x^5 - x$ is a multiple of $2 \times 2 \times 5 = 20$.

If x is even, $x-1$ and $x+1$ are both odd. For x^2+1:

$$\begin{aligned} x^2 + 1 &= (2n)^2 + 1 \\ &= 4n^2 + 1 \end{aligned}$$

which is also odd, so $x^5 - x$ has only one factor of 2, and so is not a multiple of 20. We can check this with an example. If $x = 2$,

$$\begin{aligned} x^5 - x &= 2^5 - 2 \\ &= 32 - 2 \\ &= 30 \end{aligned}$$

which is not a multiple of 20.

6.
- If $x = 7n$ then $x^7 - x$ has $x = 7n$ as a factor, so it is a multiple of 7.
- If $x = 7n+1$ then $x^7 - x$ has $(x-1) = (7n+1) - 1 = 7n$ as a factor, so it is a multiple of 7.
- If $x = 7n + 2$ then

$$\begin{aligned} x^2 + x + 1 &= (7n+2)^2 + (7n+2) + 1 \\ &= 49n^2 + 28n + 4 + 7n + 2 + 1 \\ &= 49n^2 + 35n + 7 \\ &= 7(7n^2 + 5n + 1) \end{aligned}$$

Hence as $x^7 - x$ has $(x^2 + x + 1) = 7(7n^2 + 5n + 1)$ as a factor, it is a multiple of 7.

- If $x = 7n + 3$ then

$$\begin{aligned} x^2 - x + 1 &= (7n+3)^2 - (7n+3) + 1 \\ &= 49n^2 + 42n + 9 - 7n - 3 + 1 \\ &= 49n^2 + 35n + 7 \\ &= 7(7n^2 + 5n + 1) \end{aligned}$$

Hence as $x^7 - x$ has $(x^2 - x + 1) = 7(7n^2 + 5n + 1)$ as a factor, it is a multiple of 7.

- If $x = 7n + 4$ then

$$\begin{aligned} x^2 + x + 1 &= (7n+4)^2 + (7n+4) + 1 \\ &= 49n^2 + 56n + 16 + 7n + 4 + 1 \\ &= 49n^2 + 56n + 21 \\ &= 7(7n^2 + 8n + 3) \end{aligned}$$

Hence as $x^7 - x$ has $(x^2 + x + 1) = 7(7n^2 + 8n + 3)$ as a factor, it is a multiple of 7.

- If $x = 7n + 5$ then

$$\begin{aligned} x^2 - x + 1 &= (7n+5)^2 - (7n+5) + 1 \\ &= 49n^2 + 70n + 25 - 7n - 5 + 1 \\ &= 49n^2 + 63n + 21 \\ &= 7(7n^2 + 9n + 3) \end{aligned}$$

Hence as $x^7 - x$ has $(x^2 - x + 1) = 7(7n^2 + 9n + 3)$ as a factor, it is a multiple of 7.

- If $x = 7n + 6$ then $x^7 - x$ has $(x + 1) = (7n+6) + 1 = 7n + 7 = 7(n+1)$ as a factor, so it is a multiple of 7.

Hence, $x^7 - x$ for $x > 1$ is always a multiple of 7.

Ex 12C

1.

The initial case, where $n = 1$,

$$1 = \frac{1}{2}(1)(1+1)$$

is true.

Assume the statement is true for $n = k$, i.e.

$$1 + 2 + 3 + 4 + \ldots + k = \frac{1}{2}k(k+1)$$

Then for $n = k + 1$

$$\begin{aligned} 1 + 2 + 3 + 4 + \ldots + k + (k+1) &= \frac{1}{2}k(k+1) + (k+1) \\ &= (\frac{1}{2}k + 1)(k+1) \\ &= \frac{1}{2}(k+2)(k+1) \\ &= \frac{1}{2}(k+1)\left((k+1)+1\right) \end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$.

3.

The initial case, where $n = 1$:

$$2 = 2^2 - 2$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$2 + 4 + 8 + \ldots + 2^k = 2^{k+1} - 2$$

Then for $n = k + 1$

$$\begin{aligned} 2 + 4 + 8 + \ldots + 2^k + 2^{k+1} &= 2^{k+1} - 2 + 2^{k+1} \\ &= 2(2^{k+1}) - 2 \\ &= 2^{(k+1)+1} - 2 \end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$.

2.

The initial case, where $n = 1$:

$$\begin{aligned} \text{L.H.S.} &= 1(1+1) \\ &= 2 \\ \text{R.H.S.} &= \frac{1}{3}(1+1)(1+2) \\ &= 2 \\ &= \text{L.H.S.} \end{aligned}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$1\times2+2\times3+3\times4+\ldots+k(k+1) = \frac{k}{3}(k+1)(k+2)$$

Then for $n = k+1$:

$$\begin{aligned} 1\times2+2\times3+3\times4+&\ldots+k(k+1)+(k+1)(k+2) \\ &= \frac{k}{3}(k+1)(k+2)+(k+1)(k+2) \\ &= (\frac{k}{3}+1)(k+1)(k+2) \\ &= \frac{1}{3}(k+3)(k+1)(k+2) \\ &= \frac{k+1}{3}(k+2)(k+3) \\ &= \frac{k+1}{3}\left((k+1)+1\right)\left((k+1)+2\right) \end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k+1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$.

4.

The initial case, where $n = 1$:

$$\begin{aligned} \text{L.H.S.} &= 1(1+1)^3 \\ &= 1 \\ \text{R.H.S.} &= \frac{1^2}{4}(1+1)(1+2)^2 \\ &= 1 \\ &= \text{L.H.S.} \end{aligned}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$1^3+2^3+3^3+4^3+\ldots+k^3 = \frac{k^2}{4}(k+1)^2$$

Then for $n = k+1$

$$\begin{aligned} 1^3+2^3+3^3+4^3+&\ldots+k^3+(k+1)^3 \\ &= \frac{k^2}{4}(k+1)^2+(k+1)^3 \\ &= \frac{k^2}{4}(k+1)^2+(k+1)(k+1)^2 \\ &= \frac{k^2+4(k+1)}{4}(k+1)^2 \\ &= \frac{k^2+4k+4}{4}(k+1)^2 \\ &= \frac{(k+2)^2}{4}(k+1)^2 \\ &= \frac{(k+1)^2}{4}(k+2)^2 \\ &= \frac{(k+1)^2}{4}\left((k+1)+1\right)^2 \end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k+1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$.

5.

(a) For $n = 2$, $(2n-1) = 4-1 = 3$ and $n^2 = 4$ hence

$$1+3 = 4$$

is consistent with the rule.

For $n = 3$, $(2n-1) = 6-1 = 5$ and $n^2 = 9$ hence

$$1+3+5 = 9$$

is consistent with the rule.

Verify the other statements similarly.

(b) The initial case, where $n = 1$: $2n-1 = 1$ and

$$1 = 1^2$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$1+3+5+\ldots+(2k-1) = k^2$$

Then for $n = k+1$

$$\begin{aligned} 1+3+5+\ldots+(2k-1)+&(2(k+1)-1) \\ &= k^2+(2(k+1)-1) \\ &= k^2+2k+2-1 \\ &= k^2+2k+1 \\ &= (k+1)^2 \end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k+1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$.

6.

The initial case, where $n = 1$:

$$\frac{1}{2} = \frac{2-1}{2}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}$$

Then for $n = k+1$

$$\begin{aligned}
\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} + \frac{1}{2^{k+1}} \\
&= \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}} \\
&= \frac{2(2^k - 1)}{2^{k+1}} + \frac{1}{2^{k+1}} \\
&= \frac{2(2^k - 1) + 1}{2^{k+1}} \\
&= \frac{2^{k+1} - 2 + 1}{2^{k+1}} \\
&= \frac{2^{k+1} - 1}{2^{k+1}}
\end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$.

7.

The initial case, where $n = 1$:

$$\frac{1}{1(1+1)} = \frac{1}{1+1}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Then for $n = k+1$

$$\begin{aligned}
\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2) + 1}{(k+1)(k+2)} \\
&= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2} \\
&= \frac{k+1}{(k+1)+1}
\end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$.

8.

The initial case, where $n = 1$:

$$\begin{aligned}
\text{L.H.S.} &= 1(1+2)(1+4) \\
&= 10 \\
\text{R.H.S.} &= \frac{1}{4}(1+1)(1+4)(1+5) \\
&= 10 \\
&= \text{L.H.S.}
\end{aligned}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$\begin{aligned}
1 \times 3 \times 5 + 2 \times 4 \times 6 + \ldots + k(k+2)(k+4) \\
&= \frac{k}{4}(k+1)(k+4)(k+5)
\end{aligned}$$

Then for $n = k+1$

$$\begin{aligned}
&1 \times 3 \times 5 + 2 \times 4 \times 6 + \ldots \\
&\quad + k(k+2)(k+4) + (k+1)(k+3)(k+5) \\
&= \frac{k}{4}(k+1)(k+4)(k+5) + (k+1)(k+3)(k+5) \\
&= (k+1)(k+5)\left(\frac{k}{4}(k+4) + (k+3)\right) \\
&= \frac{k+1}{4}(k+5)\left(k(k+4) + 4(k+3)\right) \\
&= \frac{k+1}{4}\left((k+1)+4\right)\left(k^2 + 4k + 4k + 12\right) \\
&= \frac{k+1}{4}\left((k+1)+4\right)\left(k^2 + 8k + 12\right) \\
&= \frac{k+1}{4}\left((k+1)+4\right)(k+2)(k+6) \\
&= \frac{k+1}{4}\left((k+1)+4\right)\left((k+1)+1\right)\left((k+1)+5\right) \\
&= \frac{k+1}{4}\left((k+1)+1\right)\left((k+1)+4\right)\left((k+1)+5\right)
\end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$.

9.

The initial case, where $n = 1$: $(x-1)$ is a factor of $x^1 - 1$ since $x - 1 = x^1 - 1$.

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$x^k - 1 = a(x-1)$$

Then for $n = k+1$

$$\begin{aligned} x^{k+1} - 1 &= x(x^k) - 1 \\ &= x(x^k - 1 + 1) - 1 \\ &= x(x^k - 1) + x - 1 \\ &= ax(x-1) + (x-1) \\ &= (ax+1)(x-1) \end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k+1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$.

10.

The initial case here is where $n = 7$, the first integer value satisfying $n > 6$:

$$\begin{aligned} \text{L.H.S.} &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \\ &= 5040 \\ \text{R.H.S.} &= 3^7 \\ &= 2187 \\ 5040 &> 2187 \end{aligned}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$; $k > 6$, i.e.

$$1 \times 2 \times 3 \times 4 \times \ldots \times k \geq 3^k$$

Then for $n = k+1$

$$1 \times 2 \times 3 \times 4 \times \ldots \times k(k+1) \geq 3^k(k+1)$$

$$3^k(k+1) = 3^{k+1}\frac{k+1}{3}$$

$$\begin{aligned} \text{Now} \quad k &> 6 \\ k+1 &> 7 \\ \frac{k+1}{3} &> 1 \\ \therefore \quad 3^k(k+1) &> 3^{k+1} \end{aligned}$$

$$\therefore \quad 1 \times 2 \times 3 \times 4 \times \ldots \times k(k+1) > 3^{k+1}$$

Thus if the statement is true for $n = k$ it is also true for $n = k+1$.

Hence since the statement is true for $n = 7$ it follows by induction that it is true for all integer $n > 6$.

11.

The initial case, where $n = 1$:

$$\begin{aligned} 7^1 + 2 \times 13^1 &= 7 + 26 \\ &= 33 \\ &= 3 \times 11 \end{aligned}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$7^k + 2 \times 13^k = 3a, \ a \in \mathrm{I}$$

Then for $n = k+1$

$$\begin{aligned} 7^{k+1} + 2 \times 13^{k+1} &= 7 \times 7^k + 13 \times 2 \times 13^k \\ &= 7 \times 7^k + (7+6) \times 2 \times 13^k \\ &= 7 \times 7^k + 7 \times 2 \times 13^k + 12 \times 13^k \\ &= 7(7^k + 2 \times 13^k) + 3(4 \times 13^k) \\ &= 7(3a) + 3(4 \times 13^k) \\ &= 3(7a + 4 \times 13^k) \end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k+1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$.

12.

The initial case, where $n = 1$:

$$\begin{aligned} \text{L.H.S.} &= 2 \\ \text{R.H.S.} &= \frac{2}{3}(1 + (-1)^{1+1}2^1 \\ &= \frac{2}{3}(1+2) \\ &= 2 \\ &= \text{L.H.S.} \end{aligned}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$2-4+8-16+\ldots+(-1)^{k+1}2^k = \frac{2}{3}(1+(-1)^{k+1}2^k)$$

Then for $n = k+1$

$$\begin{aligned} 2 - 4 + 8 - 16 + \ldots + (-1)^{k+1}2^k + (-1)^{k+2}2^{k+1} &= \frac{2}{3}(1 + (-1)^{k+1}2^k) + (-1)^{k+2}2^{k+1} \\ &= \frac{2}{3}(1 + (-1)^{k+1}2^k) + (-1)(-1)^{k+1}(2)2^k \\ &= \frac{2}{3}(1 + (-1)^{k+1}2^k) - 2(-1)^{k+1}2^k \end{aligned}$$

$$
\begin{aligned}
&= 2\left(\frac{1+(-1)^{k+1}2^k}{3} - (-1)^{k+1}2^k\right) \\
&= 2\left(\frac{1+(-1)^{k+1}2^k}{3} - \frac{3(-1)^{k+1}2^k}{3}\right) \\
&= 2\left(\frac{1+(-1)^{k+1}2^k - 3(-1)^{k+1}2^k}{3}\right) \\
&= 2\left(\frac{1-2(-1)^{k+1}2^k}{3}\right) \\
&= 2\left(\frac{1-(-1)^{k+1}2^{k+1}}{3}\right) \\
&= 2\left(\frac{1+(-1)(-1)^{k+1}2^{k+1}}{3}\right) \\
&= 2\left(\frac{1+(-1)^{(k+1)+1}2^{k+1}}{3}\right) \\
&= \frac{2}{3}(1+(-1)^{(k+1)+1}2^{k+1})
\end{aligned}
$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$.